

NON-ZERO CONSTANT CURVATURE FACTORABLE SURFACES IN PSEUDO-GALILEAN SPACE

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ABSTRACT. Factorable surfaces, i.e. graphs associated with the product of two functions of one variable, constitute a wide class of surfaces. Such surfaces in the pseudo-Galilean space with zero Gaussian and mean curvature were obtained in [1]. In this study, we provide new classification results relating to the factorable surfaces with non-zero Gaussian and mean curvature.

1. INTRODUCTION

One of challenging problems in classical differential geometry has been obtaining surfaces with prescribed Gaussian (K) and mean curvature (H). Let $z = z(x, y)$ be a real-valued function of two independent variables. In particular; for the immersed graph of z into a Euclidean space \mathbb{E}^3 , such a problem is reduced to solve the *Monge-Ampère equation* given by ([25, 28])

$$\det \left(\frac{\partial z}{\partial u_i \partial u_j} \right) = K \left(1 + |\nabla z|^2 \right)^2, \quad u_1 = x, \quad u_2 = y$$

and the *equation of mean curvature type* in divergence form

$$\operatorname{div} \left(\frac{\nabla z}{\sqrt{1 + |\nabla z|^2}} \right) = H,$$

where ∇ denotes the gradient of \mathbb{E}^2 ([18, 26, 27]). These equations are also related to the branches such as economics, meteorology, oceanography etc. [4]-[8].

Recall that the graph surfaces are also known as *Monge surfaces* (see [14], p. 398). In this study, we deal with a special Monge surface, namely *factorable surface* that is graph of the function $z(x, y) = f(x)g(y)$. Such surfaces with $K, H = \text{const.}$ in various ambient spaces have been classified in [3, 13, 15, 17, 19, 29, 32, 33]. Our purpose is to analyze the factorable surfaces in the pseudo-Galilean space \mathbb{G}_3^1 that is one of real Cayley-Klein spaces (for details, see [12, 16, 24, 30]). As distinct from the other ambient spaces, there exist two different kinds of factorable surfaces arising from the absolute figure of \mathbb{G}_3^1 . Explicitly, a Monge surface in \mathbb{G}_3^1 is said to be *factorable* if it is given in one of the explicit forms

$$\Omega_1 : z(x, y) = f(x)g(y) \quad \text{and} \quad \Omega_2 : x(y, z) = f(y)g(z).$$

We call Ω_1 and Ω_2 the *factorable surface of first* and *second kind*, respectively. Note that these surfaces have different geometric structures in \mathbb{G}_3^1 (such as metric, curvature etc.). Flat and minimal ($K, H = 0$) factorable surfaces in \mathbb{G}_3^1 were presented

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in [1]. Still, it is an open problem to obtain such surfaces with $K, H = \text{const.} \neq 0$. The present paper is devoted to solve this problem.

2. PRELIMINARIES

In this section, some basics of the Galilean geometry shall be provided from [2], [9]-[11], [20, 21, 31]. In particular, the local theory of immersed surfaces into a pseudo-Galilean space was well-structured in [22].

Let $P_3(\mathbb{R})$ denote the projective 3-space and $(u_0 : u_1 : u_2 : u_3)$ the homogeneous coordinates in $P_3(\mathbb{R})$. The *pseudo-Galilean 3-space* \mathbb{G}_3^1 is a metric space constructed within $P_3(\mathbb{R})$ having the absolute figure $\{\sigma, l, \epsilon\}$, where σ implies the *absolute plane* of \mathbb{G}_3^1 , l *absolute line* in σ and ϵ is the *hyperbolic involution* of the points of l . These arguments are given by $\sigma : u_0 = 0$, $l : u_0 = u_1 = 0$ and

$$\epsilon : (u_0 : u_1 : u_2 : u_3) \mapsto (u_0 : u_1 : u_3 : u_2).$$

The affine model of \mathbb{G}_3^1 can be introduced by changing homogenous coordinates with affine coordinates:

$$(u_0 : u_1 : u_2 : u_3) = (1 : x : y : z).$$

In terms of the affine coordinates, the *group of motions* is defined by

$$(2.1) \quad \begin{cases} x' = a_1 + x, \\ y' = a_2 + a_3x + (\cosh \theta)y + (\sinh \theta)z, \\ z' = a_4 + a_5x + (\sinh \theta)y + (\cosh \theta)z, \end{cases}$$

where a_i , $i \in \{1, \dots, 5\}$ and θ are some constants. The *pseudo-Galilean distance* is introduced with respect to the absolute figure, namely

$$d(x, y) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|}, & \text{if } x_1 = x_2, \end{cases}$$

where $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$. Note that this metric (also the absolute figure) is invariant under (2.1).

A plane is said to be *pseudo-Euclidean* if it satisfies the equation $x = \text{const.}$ Otherwise, it is called *isotropic plane*. A pseudo-Euclidean plane basically has Minkowskian metric while an isotropic plane has Galilean metric, i.e. parabolic measures of distances and angles. Contrary to its denotation, the *isotropic vectors* are contained in the pseudo-Euclidean plane $x = 0$ and, up to the induced Minkowskian metric on this plane, such vectors are categorized by their causal characters, i.e. *spacelike*, *timelike* and *lightlike*. For further details of the Minkowskian geometry, see [23].

An immersed surface into \mathbb{G}_3^1 is given by the mapping

$$r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{G}_3^1, \quad (u_1, u_2) \mapsto (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$$

and such a surface is said to be *admissible* (i.e without pseudo-Euclidean tangent plane) if $x_{,i} = \frac{\partial x}{\partial u_i} \neq 0$ for some $i = 1, 2$. The first fundamental form is given by

$$ds^2 = (\mathbf{g}_1 du_1 + \mathbf{g}_2 du_2)^2 + \omega (\mathbf{h}_{11} du_1^2 + 2\mathbf{h}_{12} du_1 du_2 + \mathbf{h}_{22} du_2^2),$$

where $\mathbf{g}_i = x_{,i}$, $\mathbf{h}_{ij} = y_{,i}y_{,j} + z_{,i}z_{,j}$, $i, j = 1, 2$, and

$$\omega = \begin{cases} 0, & \text{if } du_1 : du_2 \text{ is non-isotropic direction,} \\ 1, & \text{if } du_1 : du_2 \text{ is isotropic direction.} \end{cases}$$

A side tangent vector field in the tangent plane of the surface r is of the form $x_{,1}r_{,2} - x_{,2}r_{,1}$. Its pseudo-Galilean norm corresponds to

$$W = \sqrt{\left| (x_{,1}y_{,2} - x_{,2}y_{,1})^2 - (x_{,1}z_{,2} - x_{,2}z_{,1})^2 \right|}.$$

A surface with $W = 0$ is said to be *lightlike*. Throughout the study, all immersed admissible surfaces shall be assumed to be non-lightlike. Then the vector given by

$$S = \frac{x_{,1}r_{,2} - x_{,2}r_{,1}}{W} = \frac{1}{W} (0, x_{,1}y_{,2} - x_{,2}y_{,1}, x_{,1}z_{,2} - x_{,2}z_{,1}),$$

satisfies $S \cdot S = \varepsilon = \{-1, 1\}$, where " \cdot " denotes the Minkowskian scalar product. Hence a surface is said to be *spacelike* (*timelike*) if $\varepsilon = 1$ ($\varepsilon = -1$). The normal vector field is defined as

$$N = \frac{1}{W} (0, x_{,1}z_{,2} - x_{,2}z_{,1}, x_{,1}y_{,2} - x_{,2}y_{,1})$$

such that $N \cdot N = -\varepsilon$. The second fundamental form is $II = \sum_{i,j=1}^2 L_{ij} du_i du_j$, where if $\mathfrak{g}_1 \neq 0$

$$L_{ij} = \frac{\varepsilon}{\mathfrak{g}_1} (\mathfrak{g}_1 (0, y_{,ij}, z_{,ij}) - \mathfrak{g}_{i,j} (0, y_{,1}, z_{,1})) \cdot N,$$

otherwise

$$L_{ij} = \frac{\varepsilon}{\mathfrak{g}_2} (\mathfrak{g}_2 (0, y_{,ij}, z_{,ij}) - \mathfrak{g}_{i,j} (0, y_{,2}, z_{,2})) \cdot N$$

for $y_{,ij} = \frac{\partial^2 y}{\partial u_i \partial u_j}$, $1 \leq i, j \leq 2$. Consequently, the *Gaussian* and *mean curvature* are defined as

$$K = -\varepsilon \frac{L_{11}L_{22} - L_{12}^2}{W^2} \text{ and } H = -\varepsilon \frac{\mathfrak{g}_2^2 L_{11} - 2\mathfrak{g}_1 \mathfrak{g}_2 L_{12} + \mathfrak{g}_1^2 L_{22}}{2W^2}.$$

A surface is said to have *constant Gaussian* (resp. *mean*) *curvature* if K (resp. H) is a constant function identically. In particular, it is said to be *flat* (resp. *minimal*) if the constant function vanishes.

3. FACTORABLE SURFACES OF FIRST KIND

Let us consider the factorable surface of first kind in \mathbb{G}_3^1 given in explicit form $\Omega_1 : z(x, y) = f(x)g(y)$. Our purpose is to describe the surface whose $K = \text{const.} \neq 0$ and $H = \text{const.} \neq 0$. For this, firstly we can give the following result:

Theorem 3.1. *Let a factorable surface of first kind in \mathbb{G}_3^1 have non-zero constant Gaussian curvature K_0 . Then we have:*

$$z(x, y) = \pm \tanh \left(\sqrt{|K_0|} x + \lambda_1 \right) (y + \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Proof. Assume that Ω_1 has non-zero constant Gaussian curvature K_0 . Hence, we get a relation as follows:

$$(3.1) \quad K_0 = \frac{f g f'' g'' - (f' g')^2}{\left[1 - (f g')^2 \right]^2},$$

where $f' = \frac{df}{dx}$, $g' = \frac{dg}{dy}$, etc. K_0 vanishes identically when f or g is a constant function. Then f and g must be non-constant functions. We distinguish two cases for the equation (3.1):

Case a. $f' = f_0$, $f_0 \in \mathbb{R} - \{0\}$. Thereby (3.1) turns into the following polynomial equation on (g') :

$$K_0 + (f_0^2 - 2K_0f^2)(g')^2 + K_0f^4(g')^4 = 0,$$

which yields a contradiction.

Case b. $f'' \neq 0$. We have again two cases:

Case b.1. $g' = g_0$, $g_0 \in \mathbb{R} - \{0\}$. Then (3.1) leads to

$$(3.2) \quad \pm \sqrt{|K_0|} = \frac{g_0 f'}{1 - (g_0 f)^2}.$$

After solving (3.2), we obtain

$$f(x) = \pm \frac{1}{g_0} \tanh\left(\sqrt{|K_0|}x + \lambda_1\right), \quad \lambda_1 \in \mathbb{R}.$$

Case b.2. $g'' \neq 0$. Then (3.1) can be arranged as the following:

$$(3.3) \quad \frac{K_0 [1 - (fg')^2]^2}{ff''(g')^2} = \frac{gg''}{(g')^2} - \frac{(f')^2}{ff''}.$$

The partial derivative of (3.2) with respect to x and y leads to a polynomial equation on (g') :

$$(3.4) \quad -\left(\frac{1}{ff''}\right)' + \left(\frac{f^3}{f''}\right)'(g')^4 = 0.$$

Since all coefficients must vanish in (3.4), the contradiction $f' = 0$ is obtained. Therefore the proof is completed. \square

Theorem 3.2. *Let a factorable surface of first kind in \mathbb{G}_3^1 have non-zero constant mean curvature H_0 . Then the following occurs:*

$$z(x, y) = f_0 g(y) = \frac{1}{2H_0} \sqrt{(2H_0 y + \lambda_1)^2 \pm 1 + \lambda_2},$$

where "±" happens plus (resp. minus) when the surface is timelike (resp. spacelike). Further, f_0 is non-zero constant and λ_1, λ_2 some constants.

Proof. Relating to the mean curvature, we get

$$(3.5) \quad H_0 = \frac{fg''}{2 \left| 1 - (fg')^2 \right|^{\frac{3}{2}}}.$$

It is clear from (3.5) that g is a non-linear function. By taking partial derivative of (3.5) with respect to x , we deduce

$$(3.6) \quad f' \left| 1 - (fg')^2 \right| - \frac{3f}{2} \left| -2ff'(g')^2 \right| = 0,$$

which yields two cases:

Case a. $f = f_0 \neq 0$, $f_0 \in \mathbb{R}$, is a solution for (3.6). If the surface is spacelike, then (3.5) turns to

$$(3.7) \quad 2H_0 = \frac{f_0 g''}{\left[1 - (f_0 g')^2 \right]^{\frac{3}{2}}}.$$

By solving (3.7), we find

$$g(y) = \frac{1}{f_0 H_0} \sqrt{(2H_0 y + \lambda_1)^2 - 1 + \lambda_2},$$

where λ_1 and λ_2 are some constants. Otherwise, i.e. timelike situation yields

$$(3.8) \quad 2H_0 = \frac{f_0 g''}{\left[(f_0 g')^2 - 1\right]^{\frac{3}{2}}}.$$

After solving (3.8), we obtain

$$g(y) = \frac{1}{f_0 H_0} \sqrt{(2H_0 y + \lambda_3)^2 + 1 + \lambda_4}$$

for some constants λ_3, λ_4 .

Case b. $f' \neq 0$. If the surface is spacelike or timelike, then (3.6) implies

$$1 + 2(fg')^2 = 0,$$

which is not possible.

□

4. FACTORABLE SURFACES OF SECOND KIND

As in previous section, by assuming $K = \text{const.} \neq 0$ and $H = \text{const.} \neq 0$, we try to describe the factorable graph surfaces of second kind in \mathbb{G}_3^1 given in explicit form $\Omega_2 : x(y, z) = f(y)g(z)$. Therefore the following non-existence result can be stated:

Theorem 4.1. *There does not exist a factorable surface of second kind in \mathbb{G}_3^1 having non-zero constant Gaussian curvature.*

Proof. It is proved by contradiction. Then we suppose that Ω_2 has the Gaussian curvature $K_0 \neq 0$ in \mathbb{G}_3^1 . By a calculation, relating to the Gaussian curvature, we get

$$(4.1) \quad K_0 = \frac{fgf''g'' - (f'g')^2}{\left[(fg')^2 - (f'g)^2\right]^2},$$

where $f' = \frac{df}{dy}$, $g' = \frac{dg}{dz}$ and so on. Hereinafter f and g must be non-constant functions so that K_0 does not vanish. Point that the roles of f and g are symmetric and it is sufficient to discuss the cases depending on f . Thus, if $f'' = 0$ i.e. $f' = f_0 \neq 0$, then (4.1) turns a polynomial equation on (f) :

$$(4.2) \quad \left[K_0 (g')^4\right] f^4 - \left[2K_0 (f_0 g g')^2\right] f^2 + (f_0 g)^4 + (f_0 g')^2 = 0.$$

The fact that the coefficients must be zero yields the contradiction $g' = 0$. Hence f is a non-linear function and, by symmetry, so is g . By dividing (4.1) with $ff''(g')^2$, we can write

$$(4.3) \quad K_0 \left[\frac{f^3}{f''} (g')^2 - 2 \frac{f(f')^2}{f''} g^2 + \frac{(f')^4}{ff''} \left(\frac{g^2}{g'} \right)^2 \right] = \frac{gg''}{(g')^2} - \frac{(f')^2}{ff''}.$$

Put $f' = p$, $\dot{p} = \frac{dp}{df} = \frac{f''}{f'}$ and $g' = r$, $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$ in (4.3). Then the partial derivative of (4.3) with respect to g gives

$$(4.4) \quad K_0 \left[2 \frac{f^3}{p\dot{p}} r\dot{r} - 4 \frac{fp}{\dot{p}} g + 2 \frac{p^3}{f\dot{p}} \left(\frac{g^2}{r} \right) \left\{ \frac{d}{dg} \left(\frac{g^2}{r} \right) \right\} \right] = \frac{d}{dg} \left(\frac{g\dot{r}}{r} \right).$$

The partial derivative of (4.4) with respect to f yields

$$(4.5) \quad r\dot{r} \left[\frac{d}{df} \left(\frac{f^3}{p\dot{p}} \right) \right] - 2g \left[\frac{d}{df} \left(\frac{fp}{\dot{p}} \right) \right] + \left[\frac{d}{df} \left(\frac{p^3}{f\dot{p}} \right) \right] \left[\left(\frac{g^2}{r} \right) \frac{d}{dg} \left(\frac{g^2}{r} \right) \right] = 0.$$

By dividing (4.5) with g and taking partial derivative with respect to g , we derive

$$(4.6) \quad \underbrace{\left[\frac{d}{df} \left(\frac{f^3}{p\dot{p}} \right) \right]}_{F_1(f)} \underbrace{\left[\frac{d}{dg} \left(\frac{r\dot{r}}{g} \right) \right]}_{G_1(g)} + \underbrace{\left[\frac{d}{df} \left(\frac{p^3}{f\dot{p}} \right) \right]}_{F_2(f)} \underbrace{\left[\frac{d}{dg} \left\{ \left(\frac{g}{r} \right) \frac{d}{dg} \left(\frac{g^2}{r} \right) \right\} \right]}_{G_2(g)} = 0.$$

We have to distinguish several cases:

Case a. $F_1 = 0$. Then $f^3 = \lambda_1 p\dot{p}$, $\lambda_1 \in \mathbb{R}$, $\lambda_1 \neq 0$. We have again two cases:

Case a.1. $F_2 = 0$, namely $p^3 = \lambda_2 f\dot{p}$, $\lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$. Considering these in (4.5) implies $fp = \lambda_3 \dot{p}$, $\lambda_3 \in \mathbb{R}$, $\lambda_3 \neq 0$. Substituting these into (4.3) yields

$$(4.7) \quad K_0 \left[\lambda_1 r^2 - 2\lambda_3 g^2 + \lambda_2 \left(\frac{g^2}{r} \right)^2 \right] - \frac{g\dot{r}}{r} = \frac{-\lambda_2}{p^2}.$$

The left side of (4.7) is either a function of g or a constant, however other side is a non-constant function of f . This is not possible.

Case a.2. $G_2 = 0$. It implies $\frac{d}{dg} \left(\frac{g^2}{r} \right) = \frac{\lambda_4 r}{g}$, $\lambda_4 \in \mathbb{R}$. By considering this one into (4.5) together with the assumption of Case a, we conclude

$$(4.8) \quad \left[\frac{d}{df} \left(\frac{p}{f} \right) \right] \left[1 - \lambda_4 \left(\frac{p}{f} \right)^2 \right] = 0.$$

If $p = \lambda_5 f$, $\lambda_5 \in \mathbb{R}$, $\lambda_5 \neq 0$, in (4.8) then we have $\dot{p} = \lambda_5$. Combining it with the assumption of Case a gives $f^2 = \lambda_1 \lambda_5^2$ that contradicts with $K_0 \neq 0$.

Case b. $G_1 = 0$. Hence $r\dot{r} = \lambda_1 g$, $\lambda_1 \in \mathbb{R}$, $\lambda_1 \neq 0$. We have two cases:

Case b.1. $F_2 = 0$, i.e. $p^3 = \lambda_2 f\dot{p}$, $\lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$. Then (4.5) follows

$$(4.9) \quad \left[\frac{d}{df} \left(\frac{f}{p} \right) \right] \left[\lambda_1 \left(\frac{f}{p} \right)^2 - 1 \right] = 0.$$

If $p = \lambda_3 f$, $\lambda_3 \in \mathbb{R}$, $\lambda_3 \neq 0$, in (4.8) then we get $\dot{p} = \lambda_3$. Comparing this one with the assumption of Case b.1 gives $f^2 = \lambda_2$, which is no possible since $K_0 \neq 0$.

Case b.2. $G_2 = 0$. It follows

$$(4.10) \quad \left(\frac{g}{r} \right) \frac{d}{dg} \left(\frac{g^2}{r} \right) = \lambda_4, \quad \lambda_4 \in \mathbb{R}.$$

An integration of (4.10) with respect to g gives

$$(4.11) \quad r = \frac{g^2}{\sqrt{\lambda_4 g^2 + \lambda_5}}, \quad \lambda_5 \in \mathbb{R},$$

where λ_4 and λ_5 are not equal to zero together. After taking derivative of (4.11) with respect to g and producting with r , we conclude

$$(4.12) \quad r\dot{r} = \frac{\lambda_4 g^5 + 2\lambda_5 g^3}{(\lambda_4 g^2 + \lambda_5)^2}.$$

Due to the assumption of the Case b, (4.12) turns to the following polynomial equation on g :

$$(4.13) \quad (\lambda_4 - \lambda_1 \lambda_4^2) g^5 + 2(\lambda_5 - \lambda_1 \lambda_4 \lambda_5) g^3 - (\lambda_1 \lambda_5^2) g = 0.$$

Since $\lambda_1 \neq 0$, we get $1 = \lambda_1 \lambda_4$ and $\lambda_5 = 0$. It follows from (4.11) that $r = (\lambda_4)^{\frac{-1}{2}} g$. Then by substituting it into (4.3), we obtain

$$(4.14) \quad K_0 \left[\frac{f^3}{\lambda_4 p \dot{p}} - 2 \frac{f p}{\dot{p}} + \lambda_4 \frac{p^3}{f \dot{p}} \right] g^2 + \frac{p}{f \dot{p}} - 1 = 0.$$

This polynomial equation leads to

$$(4.15) \quad f \dot{p} = p$$

and

$$(4.16) \quad \frac{f^3}{\lambda_4 p \dot{p}} - 2 \frac{f p}{\dot{p}} + \lambda_4 \frac{p^3}{f \dot{p}} = 0.$$

Substituting (4.15) into (4.16) gives $p = \pm (\lambda_4)^{\frac{-1}{2}} f$ or $f(y) = \lambda_6 \exp(\pm y)$, $\lambda_6 \in \mathbb{R}$, $\lambda_6 \neq 0$. Further, since $r = (\lambda_4)^{\frac{-1}{2}} g$, we have $g(z) = \lambda_7 \exp(z)$, $\lambda_7 \in \mathbb{R}$, $\lambda_7 \neq 0$. However these lead the surface to be flat, i.e. $K_0 = 0$, which is not our case.

Case c. $F_1 G_1 \neq 0$. Then (4.6) can be rewritten as

$$(4.17) \quad \frac{F_1(f)}{F_2(f)} + \frac{G_2(g)}{G_1(g)} = 0,$$

which implies

$$(4.18) \quad \frac{f^3}{p \dot{p}} = \lambda_1 \frac{p^3}{f \dot{p}} + \lambda_2, \quad \text{and} \quad \left(\frac{g}{r} \right) \frac{d}{dg} \left(\frac{g^2}{r} \right) = -\lambda_1 \frac{r \dot{r}}{g} + \lambda_3,$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\lambda_1 \neq 0$. Substituting (4.18) into (4.5) gives

$$(4.19) \quad 2 \frac{f p}{\dot{p}} = \lambda_3 \frac{p^3}{f \dot{p}} + \lambda_4, \quad \lambda_4 \in \mathbb{R}.$$

Comparing (4.19) with the first equality in (4.18) leads to

$$(4.20) \quad \begin{cases} f^4 - \lambda_1 p^4 = \lambda_2 f p \dot{p} \\ 2 f^2 p^2 - \lambda_3 p^4 = \lambda_4 f p \dot{p}. \end{cases}$$

It is clear that λ_3 and λ_4 cannot be zero together. Without lose of generality, we may assume $\lambda_4 = 1$. Then (4.20) turns to

$$(4.21) \quad f^4 - 2\lambda_2 f^2 p^2 + (\lambda_2 \lambda_3 - \lambda_1) p^4 = 0$$

or

$$(4.22) \quad \left(\frac{f}{p}\right)^2 + (\lambda_2\lambda_3 - \lambda_1) \left(\frac{f}{p}\right)^{-2} = 2\lambda_2.$$

Taking derivative of (4.22) with respect to f leads to

$$(4.23) \quad \frac{d}{df} \left(\frac{f}{p}\right) \left[1 - (\lambda_2\lambda_3 - \lambda_1) \left(\frac{f}{p}\right)^{-4}\right] = 0.$$

If the ratio f/p is constant, i.e. $p = \lambda_5 f$, $\lambda_5 \in \mathbb{R}$, $\lambda_5 \neq 0$, then the second equality in (4.20) implies

$$(2\lambda_5^2 - \lambda_3\lambda_5^4) f^4 = \lambda_5^2 f^2,$$

which is not possible. This completes the proof. \square

Theorem 4.2. *Let a factorable surface of second kind in \mathbb{G}_3^1 have non-zero constant mean curvature H_0 . Then we have:*

$$x(y, z) = \lambda_1 \exp \left(\lambda_2 y + \frac{\lambda_2}{2H_0} \sqrt{(2H_0 z + \lambda_3)^2 \pm 1} \right),$$

where "±" happens plus (resp. minus) when the surface is timelike (resp. spacelike). Further, λ_1, λ_2 are non-zero constants and λ_3 some constant.

Proof. The proof is only done for spacelike situation since the calculations are almost same for other situation. Then we have

$$(fg')^2 - (f'g)^2 > 0$$

for all pairs (y, z) . Since the mean curvature is constant $H_0 \neq 0$, by a calculation, we deduce

$$(4.24) \quad 2H_0 \left[(fg')^2 - (f'g)^2 \right]^{\frac{3}{2}} = (fg')^2 f''g - 2fg(f'g')^2 + (f'g)^2 fg''.$$

Note that f is not a constant function since $H_0 \neq 0$ and, by symmetry, neither is g . Then dividing (4.24) with $fg(f'g')^2$ yields

$$(4.25) \quad 2H_0 \left[\left(\frac{f}{f'}\right)^{\frac{4}{3}} \left(\frac{g'}{g}\right)^{\frac{2}{3}} - \left(\frac{f'}{f}\right)^{\frac{2}{3}} \left(\frac{g}{g'}\right)^{\frac{4}{3}} \right]^{\frac{3}{2}} = \frac{ff''}{(f')^2} + \frac{gg''}{(g')^2} - 2.$$

Let us put $f' = p$, $\dot{p} = \frac{dp}{df} = \frac{f''}{f'}$ and $g' = r$, $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$ in (4.25). Thus (4.25) can be rewritten as

$$(4.26) \quad 2H_0 \left[\left(\frac{f}{p}\right)^{\frac{4}{3}} \left(\frac{r}{g}\right)^{\frac{2}{3}} - \left(\frac{p}{f}\right)^{\frac{2}{3}} \left(\frac{g}{r}\right)^{\frac{4}{3}} \right]^{\frac{3}{2}} = \frac{f\dot{p}}{p} + \frac{g\dot{r}}{r} - 2.$$

The partial derivative of (4.26) with respect to f gives

$$(4.27) \quad 2H_0 \left[\left(\frac{f}{p}\right)^{\frac{4}{3}} \left(\frac{r}{g}\right)^{\frac{2}{3}} - \left(\frac{p}{f}\right)^{\frac{2}{3}} \left(\frac{g}{r}\right)^{\frac{4}{3}} \right]^{\frac{1}{2}} \left[2 \left(\frac{r}{g}\right)^{\frac{2}{3}} + \left(\frac{p}{f}\right)^2 \left(\frac{g}{r}\right)^{\frac{4}{3}} \right] \frac{d}{df} \left(\frac{f}{p}\right) = \frac{d}{df} \left(\frac{f\dot{p}}{p}\right) \left(\frac{p}{f}\right)^{\frac{1}{3}}.$$

If $\dot{p} = 0$, then (4.27) reduces to

$$2 + \left(\frac{p}{f}\right)^2 \left(\frac{g}{r}\right)^2 = 0,$$

which is not possible. Thus p is not constant function and, by symmetry, so is r .

In addition, we have to consider two cases in order to solve (4.27):

Case a. $p = \lambda_1 f$, $\lambda_1 \neq 0$, is a solution for (4.27). Substituting this one into (4.26) gives

$$2H_0 \left[\lambda_1^{-\frac{4}{3}} \left(\frac{r}{g}\right)^{\frac{2}{3}} - \lambda_1^{\frac{2}{3}} \left(\frac{g}{r}\right)^{\frac{4}{3}} \right]^{\frac{3}{2}} = \frac{g\dot{r}}{r} - 1$$

or

$$2H_0 \left[\left(\frac{r}{g}\right)^2 - \lambda_1^2 \right]^{\frac{3}{2}} = \lambda_1^2 \left[\frac{r\dot{r}}{g} - \left(\frac{r}{g}\right)^2 \right].$$

The last equality can be rearranged as

$$(4.28) \quad 2H_0 = \frac{\lambda_1^2 \left(\frac{g'}{g}\right)'}{\left[\left(\frac{g'}{g}\right)^2 - \lambda_1^2\right]^{\frac{3}{2}}}.$$

An integration of (4.28) with respect to z yields

$$2H_0 z + \lambda_2 = \frac{-\frac{g'}{g}}{\sqrt{\left(\frac{g'}{g}\right)^2 - \lambda_1^2}}$$

or

$$(4.29) \quad \frac{g'}{g} = \frac{\lambda_1 (2H_0 z + \lambda_2)}{\sqrt{(2H_0 z + \lambda_2)^2 - 1}}.$$

An again integration of (4.29) with respect to z leads to

$$g(z) = \lambda_3 \exp\left(\frac{\lambda_1}{2H_0} \sqrt{(2H_0 z + \lambda_2)^2 - 1}\right), \quad \lambda_3 \neq 0.$$

Due to the assumption of Case a, we conclude $f(y) = \lambda_4 \exp(\lambda_1 y)$, $\lambda_4 \neq 0$, which gives the assertion of the theorem.

Case b. $\frac{d}{df} \left(\frac{f}{p}\right) \neq 0$. By symmetry, we deduce $\frac{d}{dg} \left(\frac{g}{r}\right) \neq 0$. Then (4.27) can be rewritten as

$$(4.30) \quad \left[\left(\frac{r}{g}\right)^2 - \left(\frac{p}{f}\right)^2\right]^{\frac{1}{2}} \left[2 + \left(\frac{p}{f}\right)^2 \left(\frac{r}{g}\right)^{-2}\right] = \frac{\frac{d}{df} \left(\frac{f\dot{p}}{p}\right) \left(\frac{p}{f}\right)}{2H_0 \frac{d}{df} \left(\frac{f}{p}\right)}.$$

The partial derivative of (4.30) with respect to g leads to

$$(4.31) \quad 2 \left(\frac{r}{g}\right)^4 - \left(\frac{r}{g}\right)^2 \left(\frac{p}{f}\right)^2 + 2 \left(\frac{p}{f}\right)^4 = 0.$$

Since the ratios $\frac{p}{f}$ and $\frac{r}{g}$ are not constant, (4.31) presents a polynomial equation which yields a contradiction. Therefore the proof is completed. \square

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